Physics

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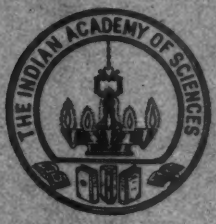
INDIAN ACADEMY OF SCIENCES

VOL. XLI] *SECTION A* [No. 6

JUNE 1955



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REMARKS ON A PROBLEM IN SYMMETRIC FUNCTIONS

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Received April 4, 1955

1. Introduction

In a former paper (*Proc. Ind. Acad. Sci.*, 1952, **35 A**, referred to hereafter as **p**) we considered the following problem: Given that the set of variables $(x_1, x_2, \ldots, x_n)$ satisfies the conditions:

$$(a)\ 0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$$

$$(b)\ \Sigma x_1 = k_1$$

$$\Sigma x_1 x_2 = k_2$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\Sigma x_1 \ldots x_{n-1} = k_{n-1}$$

where $k_1, k_2, \ldots k_{n-1}$ are fixed but unspecified constants.

(c) Maximum and minimum values of the successive x's occur alternately, *i.e.*, if x_1 has its maximum value, x_2 has its minimum value, x_3 its maximum value and so on.

To find, under these conditions, n numbers $(\alpha_1, \alpha_2 \ldots, \alpha_n)$ such that

$$0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{n-1} \leqslant x_n \leqslant \alpha_n.$$

It was shown in **p** that the α's are the roots of certain Tschebyscheff polynomials and that they possess certain symmetry properties.

The object of the present note is to offer some supplementary remarks indicating an alternative formulation of the problem as a problem of conditional maxima and minima. This formulation shows that the original problem

is but one of a class of similar problems, all of which are governed by the same extremising condition and that the "two extreme solutions" considered in the previous paper (**p**, p. 212) are indeed such. We give solutions for some of these other problems and also present some additional considerations regarding the original problem.

2. NEW FORMULATION OF THE PROBLEM

Given that

$$a_0 \leqslant x_1 \leqslant a_1 \leqslant x_2 \leqslant a_2 \ldots . \leqslant a_{n-1} \leqslant x_n \leqslant a_n \tag{1}$$

$$\left.\begin{aligned} &\Sigma x_1 = k_1 \\ &\Sigma x_1 x_2 = k_2 \\ &\ldots\ldots\ldots\ldots\ldots \\ &\Sigma x_1 x_2 \ldots . x_{n-1} = k_{n-1} \end{aligned}\right\} \tag{2}$$

To make $\phi = x_1 x_2 \ldots . x_n = k_n$ an extremum.

We assume that no two of the a's in (1) can be equal, for if two of the a's are equal, say $a_r = a_{r-1}$, then x_r would be reduced to a constant and the problem would be reduced to one in $n - 1$ variables.

By the Lagrange method of undetermined multipliers it is readily seen that the stationary values of ϕ occur when

$$\Pi (x_r - x_s) = 0 \tag{3}$$

i.e., extreme values of ϕ occur when any two of the x's are equal and, on account of (1), this can happen only when

$$x_{r-1} = x_r, \qquad r = 2, \ldots . n \tag{4}$$

Thus ϕ has $n - 1$ stationary values. It is now evident that our problem in **p** is that particular case corresponding to the maximum number of equal pairs of roots, *viz.*,

(i) $x_1 = x_2 = a_1,\ x_3 = x_4 = a_3,$ etc.

or (ii) $x_1 = a_0,\ x_2 = x_3 = a_2,\ x_4 = x_5 = a_4,$ etc.

so that ϕ has only *two* stationary values. From (3) we see that ϕ is stationary when the polynomial

$$f(x) = x^n - k_1 x^{n-1} + k_2 x^{n-2} - k_3 x^{n-3} + \ldots . \pm k_{n-1} x \mp k_n$$

has a pair of equal roots, the condition for which is the vanishing of the eliminant of

$$f\ (x) \equiv x^n - k_1 x^{n-1} + \ldots . \mp k_{n-1} x \pm k_n = 0$$

$$f'(x) \equiv n x^{n-1} - (n-1)\, k_1 x^{n-2} + \ldots . \mp k_{n-1} = 0$$

By Sylvester's theorem, the eliminant will be a relation of the form

$$\mathrm{F}(k_1,\ k_2,\ \ldots . k_{n-1},\ k_n) = 0 \tag{5}$$

in which $k_1,\ k_2,\ \ldots . k_{n-1}$ each occur to the degree n but k_n (or ϕ) occurs to the degree $n-1$, corresponding to the $n-1$ stationary values. These $(n-1)$ stationary values will in general be different, but by a proper choice of $k_1,\ k_2,\ \ldots . k_{n-1}$, the eliminant (5) can be reduced to the form

$$(\phi - \lambda)^r\ (\phi - \mu)^s = 0, \qquad r + s = n - 1 \tag{6}$$

Conversely, if (5) reduces to the form (6), the k's are determined. The two stationary values given by (6) correspond to the 'two extreme solutions' referred to above. This offers a natural approach to our original problem but it seems difficult to carry out this programme in general.

3. Other Related Problems

The conditions for an extremum in the problem considered above can also be written as follows:—

$$\left.\begin{aligned} d(\Sigma x_1) &= 0 \\ d(\Sigma x_1 x_2) &= 0 \\ \ldots\ldots\ldots\ldots\ldots & \\ d(\Sigma x_1 x_2 \ldots . x_{n-1}) &= 0 \\ d(x_1 \ldots . x_n) &= 0 \end{aligned}\right\} \tag{7}$$

the last of which holds because of the required stationary character of the function, while the remaining equations are consequences of the constancy of the concerned functions. It will be observed that *exactly the same equations are obtained if, instead of making* $\phi = k_n$ *stationary, any one of the other* k*'s is made an extremum, all the remaining* k*'s being fixed.* Thus the condition (3) for an extremum holds for all the n different problems:—

(i) any one k_r = extremum, $r = 1,\ 2, \ldots$ or n

(ii) all other k_s, $(n-1)$ in number, $(s \neq r)$ fixed.

As in the original problem, we consider for each of the new problems only the case corresponding to the maximum number of double roots. By way of illustration we give the solutions for $n = 3$ and $n = 4$.

The Case $n = 3$

Problem	a_0	a_1	a_2	a_3
k_1 =Stationary k_2, k_3 fixed	$\frac{m+1}{2m}$	1	m	$\frac{m(m+1)}{2}$
k_2, = Stationary k_3, k_1 fixed	$\frac{2}{m+1}$	1	m	$\frac{2m^2}{m+1}$
k_3 = Stationary k_1, k_2 fixed	$\frac{3-m}{2}$	1	m	$\frac{3m-1}{2}$

The Case $n = 4$

Problem	a_0	a_1	a_2	a_3	a_4
k_1 = Stationary k_2, k_3 and k_4 fixed	$\frac{m^2(m-1)}{(m+1)(m-2)^2}$	1	m^2-1	$\frac{m^2(m^2-1)}{4-m^2}$	$\frac{m^2(m+1)}{(m-1)(m+2)^2}$
k_2 = Stationary k_3, k_4 and k_1 fixed	$\frac{(m^2-m+1)-}{(m+1)\sqrt{m^2+1}}$	1	m	m^2	$\frac{(m^2-m+1)+}{(m+1)\sqrt{m^2+1}}$
k_3 = Stationary k_4, k_1 and k_2 fixed	a	1	m	$\frac{m(3m-1)}{m+1}$	c
	where a, c are the roots of $(m+1)^2t^2 - 2(2m+1)(m^2-1)t + (3m-1)^2 = 0$				
k_4 = Stationary k_1, k_2 and k_3 fixed	$m-(m-1)\sqrt{2}$	1	m	$2m-1$	$m+(m-1)\sqrt{2}$

In our original problem (k_n-stationary) we took $a_0 = 0$. It is readily verified that the Polynomial Identity formulated for the case $a_0 \neq 0$ transforms itself into the very identity for the case $a_0 = 0$ (**p**, p. 217, 220) by the mere substitution:

$$\gamma^v = t - a_0$$

$$a_r - a_0 = a_r'$$

$$\xi - a_0 = \gamma\xi'$$

$$\gamma \neq 0$$

Taking $\gamma = 1$, we see that, when $a_0 \neq 0$, the differences $(a_0 - a_1)$, $(a_2 - a_0)\ldots$, $(a_n - a_0)$ are the roots of the Tschebyscheff polynomial found in **p**. We might here draw attention to one important distinction between the problem of k_n stationary and the others corresponding to $k_r\,(r \neq n)$ stationary. In the former case, the ratios $(a_1 - a_0) : (a_2 - a_0) : \ldots : (a_n - a_0)$ are independent of any arbitrary parameter; this is *not* the case for the other problems.

4. Remarks on the Symmetry Property

For the solution of the original problem we established in **p** the following symmetry relations for any n:

$$a_0 + a_n = a_1 + a_{n-1} = a_2 + a_{n-2} = \cdots \tag{8}$$

It should be possible to prove these relations from the basic governing relations, without obtaining the explicit form of the solution.

Consider the case $n = 2m + 1$. We have then the polynomial identity (**p**, p. 217), *generalised* to the present case $a_0 \neq 0$,

$$\begin{aligned}\phi(t) &\equiv (t - a_0)\,(t - a_2)^2 (t - a_4)^2 \ldots (t - a_{2m})^2 \\ &\equiv (t - a_1)^2 (t - a_3)^2 \ldots (t - a_{2m-1})^2 (t - \xi) + \\ &\qquad (\xi - a_0)\,(a_1 - a_0)^2\,(a_3 - a_0)^2 \ldots (a_{2m-1} - a_0)^2.\end{aligned}$$

Since this is an identity, $a_0, a_2, a_4, \ldots a_{2m}$ as zeros of $\phi(t)$ are *uniquely* determined as functions of $a_1, a_3, \ldots a_{2m-1}, \xi$. By writing $\xi - t + a_0$ for t, the identity becomes, after a slight rearrangement

$$\begin{aligned}(t - a_0)\,[t - (\xi + a_0 - a_1)]^2\,[t - (\xi + a_0 - a_3)]^2 &\\ [t - (\xi + a_0 - a_{2m-1})]^2 &\\ = [t - (\xi + a_0 - a_2)]^2\,[t - (\xi + a_0 - a_4)]^2 &\\ [t - (\xi + a_1 - a_{2m})]^2 (t - \xi) &\\ + (\xi - a_0)\,(a_1 - a_0)^2\,(a_3 - a_0)^2 \ldots (a_{2m-1} - a_0)^2 &\end{aligned}$$

Comparing this with the previous form we can see that the identity is satisfied if we write

$$\xi + a_0 - a_1 = a_{2m}$$

$$\xi + a_0 - a_3 = a_{2m-2}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\xi + a_0 - a_{2m-1} = a_2$$

Of course, $\xi = a_{2m+1} = a_n$.

On account of the above-mentioned uniqueness, it follows that the sought relations between $(a_0, a_2, a_4 \ldots a_{2m})$ and $(a_1, a_3 \ldots a_{2m-1}, \xi)$ are just these, which are of course the desired symmetry relations. However a similar proof does not work for the case $n = 2m$.

THE GREEN'S FUNCTIONS FOR EQUATIONS OF PARTICLES OF ARBITRARY SPIN

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Received April 21, 1955

(Communicated by Dr. H. J. Bhabha, F.A.SC.)

THE wave equation for a particle of arbitrary spin can be written in the form (Bhabha, 1949)

$$(p_k\, a^k + \chi)\, \psi = 0, \tag{1}$$

where

$$p_k = -\, i\, \frac{\partial}{\partial x^k}\ ;\ \ x^0 (= t),\ x^1,\ x,^2\ x^3$$

are the co-ordinates of a point in space-time; a^k are $\nu \times \nu$ matrices, and ψ is a ν-component wave function. χ is a constant related to the masses of the particle described by (1). a^0 satisfies, in the general case, the minimal equation

$$(a^0)^m \{(a)^2 - a_1{}^2\} \{(a^0)^2 - a_2{}^2\} \ldots . \{(a^0)^2 - a_n{}^2\} = 0. \tag{2}$$

Then also

$$\mathrm{P}^m (\mathrm{P}^2 - a_1{}^2 p^2)\ (\mathrm{P}^2 - a_2{}^2 p^2) \ldots . (\mathrm{P}^2 - a_n{}^2\ p^2) = 0, \tag{3}$$

where

$$\mathrm{P} = p_k\ a^k$$

and

$$p^2 = p_k\ p^k.$$

Equation (1) will describe a particle which has states of masses χ/a_1, $\chi/a_2, \ldots, \chi/a_n$.

The commutation relations for the particle field have been found by Udgaonkar (1953). They are

$$[\psi\,(x),\ \psi^\dagger\,(x')\,\mathrm{D}] = -\, i\mathrm{S}\,(x - x'), \tag{4}$$

where

$$\mathrm{S}(x) = \sum_{r=1}^{n} \frac{(-1)^m}{\chi^{2n+m-2}\, a_r{}^2 \left(1 - \frac{a_1{}^2}{a_r{}^2}\right) \left(1 - \frac{a_2{}^2}{a_r{}^2}\right) \cdots \left(1 - \frac{a_{r-1}{}^2}{a_r{}^2}\right) \left(1 - \frac{a_{r+1}{}^2}{a_r{}^2}\right) \cdots \left(1 - \frac{a_n{}^2}{a_r{}^2}\right)}$$

$$\times \mathrm{P}^m \left(\mathrm{P}^2 - \frac{\chi^2 a_1{}^2}{a_r{}^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_{r-1}{}^2}{a_r{}^2}\right) (\mathrm{P} - \chi) \left(\mathrm{P}^2 - \frac{\chi^2 a_{r+1}{}^2}{a_r{}^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_n{}^2}{a_r{}^2}\right)$$

$$\times \Delta_r\,(x); \tag{5}$$

here $\Delta_r(x)$ is the well-known invariant function of Jordan and Pauli for mass χ/a_r. [A, B] stands for AB + BA or AB − BA according as the charge density or the energy density is positive definite.

We first prove that $S(x)$ defined by (5) satisfies

$$(P + \chi)\, S(x) = 0.$$

Now

$$(P + \chi)\, S(x) = \sum_{r=1}^{n} A_r P^m \left(P^2 - \frac{\chi^2 a_1^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_{r-1}^2}{a_r^2}\right) (P^2 - x^2)$$

$$\times \left(P^2 - \frac{\chi^2 a_{r+1}^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_n^2}{a_r^2}\right) \Delta_r(x) \tag{6}$$

where we have put

$$A_r = \frac{(-1)^m}{\chi^{2n+m-2}\, a_r^2 \left(1 - \frac{a_1^2}{a_r^2}\right)\left(1 - \frac{a_2^2}{a_r^2}\right) \cdots \left(1 - \frac{a_{r-1}^2}{a_r^2}\right)\left(1 - \frac{a_{r+1}^2}{a_r^2}\right) \cdots \left(1 - \frac{a_n^2}{a_r^2}\right)}, \tag{7}$$

Since $\Delta_r(x)$ satisfies the equation

$$\left(p^2 - \frac{\chi^2}{a_r^2}\right) \Delta_r(x) = 0, \tag{8}$$

we can replace χ^2/a_r^2 by p^2 in equation (5); we get thereby

$$(P + \chi)\, S(x) = \Sigma\, A_r P^m (P^2 - a_1^2 p^2)\, (P^2 - a_2^2 p^2) \cdots (P^2 - a_n^2 p^2)\, \Delta_r(x)$$

$$= 0;$$

the last step follows from equation (3).

Let

$$\bar{\Delta}_r(x) \equiv -\tfrac{1}{2}\, \xi(x)\, \Delta_r(x), \tag{9}$$

where

$$\xi(x) = +1 \text{ for } x_0 > 1$$

$$= -1 \text{ for } x_0 < 1.$$

Then [*cf.* Pauli, 1950]

$$\left(p^2 - \frac{\chi^2}{a_r^2}\right) \bar{\Delta}_r(x) = -\delta(x), \tag{10}$$

where

$$\delta(x) = \delta(x^0)\, \delta(x^1)\, \delta(x^2)\, \delta(x^3).$$

Let $\bar{\mathrm{S}}(x)$ denote the function obtained from $\mathrm{S}(x)$ by replacing the $\Delta_r(x)$ by $\bar{\Delta}_r(x)$, *viz.*,

$$\bar{\mathrm{S}}(x) = \Sigma \mathrm{A}_r \mathrm{P}^m \left(\mathrm{P}^2 - \frac{\chi^2 a_1^2}{a_r^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_{r-1}^2}{a_r^2}\right) (\mathrm{P} - \chi) \times \left(\mathrm{P}^2 - \frac{\chi a_{r+1}^2}{a_r^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_r^2}{a_r^2}\right) \bar{\Delta}_r(x). \tag{11}$$

We shall now prove that

$$(\mathrm{P} + \chi)\, \bar{\mathrm{S}}(x) = -(-1)^m \frac{\mathrm{P}^m}{\chi^m} \delta(x).$$

Operating on equation (11) from the left by $(\mathrm{P} + \chi)$ we get

$$\begin{aligned}(\mathrm{P} + \chi)\, \bar{\mathrm{S}}(x) &= \Sigma \mathrm{A}_r \mathrm{P}^m \left(\mathrm{P}^2 - \frac{\chi^2 a_1^2}{a_r^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_n^2}{a_r^2}\right) \bar{\Delta}_r(x) \\ &= \Sigma \mathrm{A}_r \mathrm{P}^m \left\{\mathrm{P}^{2n} + \mathrm{P}^{2(n-1)} \left(-\chi^2 \frac{a_1^2}{a_r^2} - \chi^2 \frac{a_2^2}{a_r^2} - \cdots - \frac{\chi^2 a_n^2}{a_r^2}\right)\right. \\ &\quad + \mathrm{P}^{2(n-2)} \left(\frac{\chi^2 a_1^2}{a_r^2} \frac{\chi^2 a_2^2}{a_r^2} + \frac{\chi^2 a_1^2}{a_r^2} \frac{\chi^2 a_3^2}{a_r^2} + \cdots\right) \\ &\quad \left. + \cdots + (-1)^n \left(\frac{\chi^2}{a_r^2}\right)^n a_1^2 a_2^2 \cdots a_n^2 \right\} \bar{\Delta}_r(x). \end{aligned} \tag{12}$$

Now from equation (3)

$$\begin{aligned}\mathrm{P}^m \{&\mathrm{P}^{2n} + \mathrm{P}^{2(n-1)} (-a_1^2 p^2 - a_2^2 p^2 - \cdots - a_n^2 p^2) \\ &+ \mathrm{P}^{2(n-2)} (a_1^2 p^2 a_2 p^2 + a_1^2 p^2 a_3^2 p^2 + \cdots) + \cdots \\ &+ (-1)^n p^{2n} a_1^2 a_2^2 \cdots a_n^2\} = 0.\end{aligned}$$

Substituting for P^{2n+m} from this equation in equation (12) we obtain

$$\begin{aligned}(\mathrm{P} + x)\, \bar{\mathrm{S}}(x) = \Sigma \mathrm{A}_r \mathrm{P}^m \Big[&\mathrm{P}^{2(n-1)} \left\{a_1^2 \left(p^2 - \frac{\chi^2}{a_r^2}\right) + a_2^2 \left(p^2 - \frac{\chi^2}{a_r^2}\right) + \cdots \right. \\ &\left. + a_n^2 \left(p^2 - \frac{\chi^2}{a_r^2}\right)\right\} + \mathrm{P}^{2(n-2)} \left\{a_1^2 a_2^2 \left(\frac{\chi^4}{a_r^4} - p^4\right) + \left(\frac{\chi^4}{a_r^4} - p^4\right)\right. \\ &\left. \times a_1^2 a_3^2 + \cdots\right\} + \cdots + (-1)^n a_1^2 a_2^2 \cdots \\ &a_n^2 \left\{\left(\frac{\chi^2}{a_r^2}\right)^n - p^{2n}\right\}\Big] \bar{\Delta}_r(x).\end{aligned}$$

On the right-hand side of this equation $(p^2 - \chi^2/a_r^2)$ is a common factor. In view of the inhomogeneous equation (10) satisfied by $\bar{\Delta}_r(x)$, we obtain

$$(\mathrm{P} + \chi)\,\bar{\mathrm{S}}(x) = -\sum_r \mathrm{A}_r\,\mathrm{P}^m \Big[\mathrm{P}^{2(n-1)}(a_1^2 + a_2^2 + \cdots + a_n^2) + \mathrm{P}^{2(n-2)}(-a_1^2a_2^2 - a_1^2a_3^2 - \cdots)\left(p^2 + \frac{\chi^2}{a_r^2}\right) + \cdots + (-1)^n a_1^2 a_2^2 \cdots a_n^2 \left\{-p^{2(n-1)} - p^{2(n-2)}\frac{\chi^2}{a_r^2} - p^{2(n-3)}\left(\frac{\chi^2}{a_r^2}\right)^2 - \cdots - p^2\left(\frac{\chi^2}{a_r^2}\right)^{n-2} - \left(\frac{\chi^2}{a_r^2}\right)^{n-1}\right\}\Big]\delta(x). \quad (13)$$

Now the numbers A_r defined by (7) satisfy the following identities [Bhabha, 1950; *cf.* also the Appendix]:

$$\sum_{r=0}^{n} \mathrm{A}_r = 0; \quad \sum \frac{1}{a_r^2}\mathrm{A}_r = 0; \quad \cdots \sum \frac{1}{a_r^{2(n-2)}}\mathrm{A}_r = 0; \quad (14\,a)$$

$$\sum \frac{1}{a_r^{2(n-1)}}\mathrm{A}_r = (-1)^{m+n-1}\,\frac{1}{\chi^{2n+m-2}}\,\frac{1}{a_1^2\,a_2^2 \cdots a_r^2}. \quad (14\,b)$$

Hence on the right-hand side of equation (13) all terms, excepting the last one which contains $(\chi^2/a_r^2)^{n-1}$, vanish. The contribution due to this term is

$$-(-1)^m \frac{\mathrm{P}^m}{\chi^m}\,\delta(x)$$

so that

$$(\mathrm{P} + \chi)\,\bar{\mathrm{S}}(x) = -(-1)^m \frac{\mathrm{P}^m}{\chi^m}\,\delta(x) \quad (15)$$

We now define

$$\mathrm{G}(x) = \Big[\bar{\mathrm{S}}(x) + \frac{(-1)^m}{\chi^m}\{\mathrm{P}^{m-1} - \chi\,\mathrm{P}^{m-2} + \chi^2\,\mathrm{P}^{m-3} - \cdots + (-1)^{m-1}\chi^{m-1}\}\,\delta(x)\Big]. \quad (16)$$

Then

$$(\mathrm{P} + \chi)\,\mathrm{G}(x) = \Big[-(-1)^m \frac{\mathrm{P}^m}{\chi^m}\,\delta(x) + \frac{(-1)^m}{\chi^m}\{\mathrm{P}^m + (-1)^{m-1}\chi^m\}\,\delta(x)\Big] = -\delta(x).$$

Hence $\mathrm{G}(x)$ is the Green's function for equation (1).

The positive and negative frequency parts $S^+(x)$ and $S^-(x)$ respectively, of $S(x)$ are obtained by replacing the Δ_r in $S(x)$ by Δ_r^+ and Δ_r^-, which are defined by

$$\Delta_r^+ = -\frac{2}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\omega}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

$$\Delta_r^- = \frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\omega}\, e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)},\quad \omega = +\sqrt{\mathbf{k}^2+\frac{\chi^2}{a_r{}^2}}\,.$$

The function $S^{(1)} \equiv i(S^+ - S^-)$ is also obtained by replacing the Δ_r in S by $\Delta_r^{(1)}$. All these functions satisfy the homogeneous wave equation. The Feynman function S_F is defined by (*cf.* Pauli, 1950)

$$S_F(x) = S^{(1)}(x) - 2i\,G(x) \tag{17}$$

and satisfies the equation

$$(P + \chi)\,S_F(x) = 2i\,\delta(x). \tag{18}$$

Putting

$$\Delta_{Fr} = \Delta_r^{(1)} - 2i\,\bar{\Delta}_r \tag{19}$$

we obtain

$$S_F(x) = \sum_{r=1}^{n} A_r\, P^m \left(P^2 - \frac{\chi^2 a_1{}^2}{a_r{}^2}\right)\cdots\left(P^2 - \frac{\chi^2 a^2{}_{r-1}}{a_r{}^2}\right)(P-\chi)\left(P^2 - \frac{\chi^2 a^2{}_{r+1}}{a_r}\right)$$

$$\cdots\left(P^2 - \frac{\chi^2 a_r{}^2}{a_r{}^2}\right)\Delta_{Fr}(x)$$

$$-2i\,\frac{(-1)^m}{\chi^m}\{P^{m-1} - \chi P^{m-2} + \chi^2 P^{m-3} - \cdots + (-1)^{m-1}\chi^{m-1}\}\,\delta(x). \tag{20}$$

To get the momentum space representation of $S_F(x)$ we make use of the following Fourier representations

$$\Delta_{Fr}(x) = \frac{-2i}{(2\pi)^4}\int^{C} d^4p\, \frac{e^{ipx}}{-p^2 + \frac{\chi^2}{a_r{}^2}}, \tag{21}$$

$$\delta(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}, \tag{22}$$

$$px = p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3,$$

$$p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2.$$

In the first equation above the integration over p^1, p^2, p^3 is along the real axes, and over p^0 is along the real axis with two small detours one below

the point $-\{(p')^2+(p^2)^2+(p^3)^2+\chi^2/a_r{}^2\}^{\frac{1}{2}}$ and the other above the point $+\{(p')^2+(p^2)^2+(p^3)^2+\chi^2/a_r{}^2\}^{\frac{1}{2}}$ in the p^0 plane. We obtain

$$S_F(x) = \frac{2i}{(2\pi)^4}\int d^4p\, S_F(p)\, e^{ipx}, \tag{23}$$

where

$$S_F(p) = -\Big[\sum_r A_r P^m \Big(P - \chi^2\frac{a_1{}^2}{a_r{}^2}\Big)\cdots\Big(P^2 - \chi^2\frac{a_{r-1}{}^2}{a_r{}^2}\Big)(P-\chi)$$
$$\times\Big(P^2-\chi^2\frac{a^2{}_{r+1}}{a_r{}^2}\Big)\cdots\Big(P^2-\chi^2\frac{a_n{}^2}{a_r{}^2}\Big)\frac{1}{-p^2+\frac{\chi^2}{a_r{}^2}}$$
$$+\frac{(-1)^m}{\chi^m}\{P^{m-1}-\chi P^{m-2}+\chi^2P^{m-3}-\cdots+(-1)^{m-1}\chi^{m-1}\}\Big]. \tag{24}$$

Making use of the identities (14) it can easily be proved that

$$(P+\chi)\ S_F(P) = 1,$$

so that

$$S_F(x) = \frac{2i}{(2\pi)^4}\int d^4p\,(P+\chi)^{-1}\, e^{ipx}. \tag{25}$$

It should be noted that this is only a formal way of writing $S_F(x)$, since $(P+\chi)$ is a singular operator for certain values of p_k, because $(P+\chi)\,\psi = 0$ for non-vanishing ψ, and $(P+\chi)^{-1}$ then does not exist. At the poles of $(P+\chi)^{-1}$ the path of integration in (25) has to be chosen as stated above.

The function $S_F(x)$ given above is twice the propagation function K_+ as defined in the theory of Feynman (1949). Hence the graphical techniques of Feynman and Dyson can be applied in the general case of particles of arbitrary spins if one uses the function $\frac{1}{2}S_F(x)$ for the internal particle lines.

In conclusion, I should like to express my sincere thanks to Professor H. J. Bhabha for his interest in the work.

APPENDIX

The following proof of identities (14 *a*, *b*) is due to Dr. K. G. Ramanathan:

Let

$$f(x) = (x - a_1^2)\ (x - a_2^2) \cdots (x - a_n^2) \tag{A 1}$$

and let $\phi(x)$ be a polynomial of degree $< n$. Then

$$\frac{\phi(x)}{f(x)} = \frac{A_1}{x - a_1^2} + \frac{A_2}{x - a_2^2} + \cdots + \frac{A}{x - a_n^2}, \tag{A 2}$$

where

$$A_r = \frac{\phi(a_r^2)}{(a_r^2 - a_1^2)(a_r^2 - a_2^2) \cdots (a_r^2 - a_{r-1}^2)(a_r^2 - a_{r+1}^2) \cdots (a_r^2 - a_n^2)}. \tag{A 3}$$

Let

$$\phi(x) = x^{n-2}$$

then

$$x^{n-2} = \sum_{r=1}^{n} \frac{a_r^{2(n-2)}}{(a_r^2 - a_1^2) \cdots (a_r^2 - a_{r-1}^2)(a_r^2 - a_{r+1}^2) \cdots (a_r^2 - a_n^2)} \times (x - a_1^2)(x - a_2^2) \cdots (x - a_{r-1}^2)(x - a_{r+1}^2) \cdots (x - a_n^2). \tag{A 4}$$

Equating the coefficient of x^{n-1} on the right-hand side of this equation to zero we get

$$\Sigma \frac{a_r^{2(n-2)}}{(a_r^2 - a_1^2)(a_r^2 - a_2^2) \cdots (a_r^2 - a_{r-1}^2)(a_r^2 - a_{r+1}^2) \cdots (a_r^2 - a_n^2)} = 0. \tag{A 5}$$

This is the first of the identities (14 *a*) of the text. Similarly, taking $\phi(x) = x^{n-3}, x^{n-4}, \ldots, x$, we obtain other identities of set (14 *a*).

To prove (14 *b*) take $\phi(x) = 1$. Then

$$1 = \Sigma \frac{1}{(a_r^2 - a_1^2) \cdots (a_r^2 - a_{r-1}^2)(a^2 - a_{r+1}^2) \cdots (a_r^2 - a_n^2)} \times (x - a_1^2)(x - a_2^2) \cdots (x - a_{r-1}^2)(x - a_{r+1}^2) \cdots (x - a_n^2). \tag{A 6}$$

The constant term on the right-hand side must equal 1. Hence

$$1 = (-1)^{n-1} \sum_r \frac{a_1^2 a_2^2 \cdots a_n^2}{a_r^2 (a_r^2 - a_1^2) \cdots (a_r^2 - a_{r-1}^2)(a_r^2 - a_{r+1}^2) \cdots (a_r^2 - a_n^2)}, \quad \text{(A 7)}$$

which is the identity (14 *b*).

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THERMAL EXPANSION OF POTASSIUM NITRATE

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Received April 26, 1955
(Communicated by G. N. Ramachandran, F.A.SC.)

Introduction

In a previous report (Lonappan, 1955) was described an investigation of the thermal expansion of potassium chlorate by an X-ray method. For this monoclinic crystal two of the principal expansion coefficients were small, of the order of 30 to 40×10^{-6}, while the third one was much larger, about 130×10^{-6}. The direction of the latter was almost perpendicular to the plane of the O_3 groups of the chlorate ions, all of which are parallel in the structure of the crystal. It was thought worthwhile to extend the study to other crystals having similar O_3 groups, but having a different symmetry. Studies have been made with potassium nitrate (KNO_3) which is orthorhombic and orthoboric acid (H_3BO_3) which is triclinic. The results with potassium nitrate are reported here; those with boric acid will be discussed in a separate paper.

Potassium nitrate crystallizes in the class *mmm* and belongs to the space group V_h^{16} — P*mcn*. The unit cell dimensions are $a = 5{\cdot}43$ Å; $b = 9{\cdot}17$ Å; $c = 6{\cdot}45$ Å and there are four molecules per unit cell (Edwards, 1931). Since the crystal is orthorhombic the principal axes of the ellipsoid of expansion coincide with the crystallographic axes. The present study of thermal expansion has been made with a good degree of accuracy for a range of temperature 30 to 100° C. The study could not be extended for higher ranges of temperature, as in the case of potassium chlorate, since potassium nitrate undergoes a transformation at about 128° C., the crystal changing over to the rhombohedral form (Groth, 1910; Kracek, Barth and Kosanda, 1932) and the crystal, though retaining its external shape, gets broken into fine crystallites (Leonhardt and Borchert, 1936) as exhibited by the powder lines appearing in the rotation photograph above this temperature. The complete ellipsoid of expansion could be determined by finding the variations in the spacings of the various reflections in [100] and [001] zones using appropriate crystals.

2. EXPERIMENTAL DETAILS

Rotation photographs of the zero layer were taken for [100] and [001] zones using the 19 cm. camera (Unicam) and the spacings of the high angle reflections at the two temperatures were accurately determined using the 85° fiducial mark in the camera, as described earlier (Lonappan, 1955). The overlap of reflections was avoided by the use of the 'tilted crystal method'. The indices of the reflections determined by the reciprocal lattice diagram were confirmed by taking Weissenberg photographs of the crystal. The thermal expansion perpendicular to a number of crystallographic planes in a particular zone were thus determined. From these the principal expansion coefficients were calculated. Only reflections having a θ-value $> 62°$ were used for the calculation of thermal expansion. Five such high-angle reflections were obtained in each zone.

From a knowledge of the indices of the planes, the values of the inclination ϕ of their normals to a chosen direction (b-axis) in the zone can be calculated. Then the relation between α_ϕ, the thermal expansion along any direction and the inclination ϕ of the direction concerned with b-axis takes the following form

$$\text{For the [100] zone, } \alpha_\phi = \alpha_b + \sin^2\phi\,(\alpha_c - \alpha_b) \tag{1}$$

$$\text{For the [001] zone, } \alpha_{\phi'} = \alpha_b + \sin^2\phi'\,(\alpha_a - \alpha_b) \tag{2}$$

There are five equations of type (1) and five equations of type (2) from which α_a, α_b and α_c are to be determined. Considering α as a function of ϕ and ϕ' three normal equations were formed using the ten equations, giving suitable weights to the α values according to their proximity to 90°. The probable errors were calculated using standard methods (Whittaker and Robinson, 1944).

As mentioned earlier, potassium nitrate undergoes a transformation at about 128° C. On heating single crystals of potassium nitrate above this temperature in the camera, the crystal became polycrystalline even when heated very slowly through the transition temperature and consequently the present method could not be used to study the thermal expansion for higher temperatures. The powder lines which appeared in the rotation photographs about the [100] and [001] axes taken at 200° C. were measured and could be indexed on the basis of the rhombohedral unit cell of KNO_3 (I), $a = 4{\cdot}50$ Å; $\alpha = 73°30'$ (Kracek, Barth and Kosanda, 1932). In both cases the crystals were found to give only powder lines when slowly cooled to room temperature and these corresponded to those given by the normal orthorhombic structure KNO_3 (II). The third modification of potassium

nitrate KNO_3 (III) reported to occur when KNO_3 is quenched from a high temperature was not observed in these experiments.

3. Results of the Investigation

The measured values of the expansion coefficient perpendicular to various planes in the two zones studied are given in Table I. The principal expan-

TABLE I

Measured thermal expansion coefficients of reflections in the [001] *and* [100] *zones*

Indices of reflection	Inclination with b^*-axis	$\sin^2\phi$	Thermal expansion $\alpha_\phi \times 10^6$
6 6 0	59° 22′	0·7404	23·05
1 11 0	8° 44′	0·0230	21·37
5 7 0	50° 20′	0·5925	23·19
6 4 0	68° 27′	0·8651	23·34
2 10 0	18° 40′	0·1024	21·70
0 3 8	75° 14′	0·9350	171·73
0 8 6	46° 50′	0·5319	107·71
0 2 8	80° 2′	0·9700	177·85
0 10 4	29° 38′	0·2445	61·33
0 9 5	38° 18′	0·3841	84·52

sion coefficients along the three crystallographic axes, calculated from these data, are

$$\alpha_{11}\ (b\text{-axis}) = 22{\cdot}0 \pm 0{\cdot}3;$$
$$\alpha_{22}\ (a\text{-axis}) = 23{\cdot}5 \pm 0{\cdot}5;$$
$$\alpha_{33}\ (c\text{-axis}) = 182{\cdot}6 \pm 0{\cdot}2; \qquad \times 10^{-6}$$

Fig. 1 brings out the agreement between the observed values and those calculated from the constants of the ellipsoid of expansion. In Fig. 1, α is plotted against $\sin^2\phi$ or $\sin^2\phi'$ as the case may be. The observed values

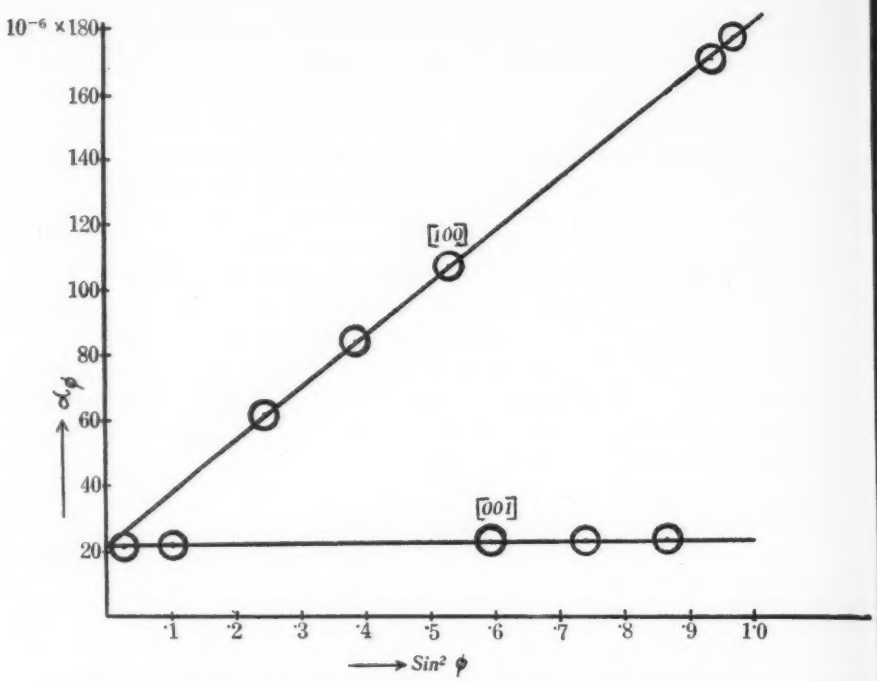

FIG. 1. Variation of expansion coefficient (30 to 100° C.) with $\sin^2 \phi$ in the two zones [100] and [001]. The circles represent the experimental values and the straight lines have been drawn using the calculated values of α_a, α_b and α_c.

should lie on two straight lines and they are found to be very close to the theoretical values.

It is interesting that α_a and α_b are found to be almost equal whereas α_c is very much higher. This is due to the fact that the planes of all the nitrate ions occurring in the crystal are parallel to one another. According to the structure determined by Edwards, the O_3 groups all lie in the *c*-plane. In the present study the maximum value of thermal expansion coefficient is obtained along the *c*-axis, which is perpendicular to the plane of the O_3 group, as it should be. The similarity between the morphology and structure of potassium nitrate and aragonite has already been pointed out by Miller (1840) and Edwards (1931). However the actual values of the thermal expansion coefficients for aragonite are 10, 16 and 33×10^{-6} while for KNO_3 they are 22, 23·5 and 183×10^{-6}. It will be noticed that the value of the largest coefficient is very different in the two cases. This is not surprising for a similar large difference occurs even between calcite and $NaNO_3$ (Table II) which

are closely related in structure. It appears that the coefficient normal to the plane of the ions is much larger with nitrates than with carbonates.

TABLE II

Thermal expansion of some crystals of the type ABO_3

Crystal	Thermal expansion $\times 10^6$		
	α_{11}	α_{22}	α_{33}*
Aragonite	10	16	33
Calcite	− 6	− 6	25
$NaNO_3$	11	11	120
$KClO_3$	30	39	134
KNO_3	22	23·5	183

* α_{33} always normal to plane of O_3 group.

Coefficient of *cubical expansion* calculated from the present data namely $\alpha_{cub} = (\alpha_{11} + \alpha_{22} + \alpha_{33}) = 228 \times 10^{-6}$ agrees with the directly measured value of 211×10^{-6} reported by Bellati and Finazzi (1910).

The value of cubical expansion coefficient may be used to verify the Grüneisen relation for potassium nitrate. The density of the crystal at room temperature is 2·098, from which the molar volume V_0 is calculated to be 48·2. The molar specific heat (C) as given in the *Landolt-Bornstein Tabellen* is 24·15. Unfortunately the value of the compressibility K_0 required in the relation

$$\gamma = V_0 \alpha_{cub} / K_0 C_v$$

(where γ is the Grüneisen constant) is not available in the literature. However γ may be evaluated from the formula

$$\gamma = -\frac{\partial (\log \nu)}{\partial (\log V)} = -\frac{\frac{1}{\nu}\frac{\partial \nu}{\partial T}}{\frac{1}{V}\frac{\partial V}{\partial T}} = \frac{\chi_\nu}{\alpha_{cub}}$$

Here χ_ν is the proportional rate of change of frequency with temperature, as defined by Ramachandran (1947 *a*). The value of χ_ν for potassium nitrate is not available, but the corresponding value of sodium nitrate may be calculated from the Raman Effect data for this crystal (Nedungadi, 1939).

The tilting and transverse oscillations of NO_3 group in sodium nitrate have χ-values of 800 and 600×10^{-6} respectively (Ramachandran, 1947 *b*), the mean of which, namely 7×10^{-4} may be taken for our calculations. This yields the value 3·07 for the Grüneisen constant.

Putting this value of γ in the equation mentioned earlier, we get

$$K_0 = 35\cdot28\times10^{-13}$$

This value for compressibility is of the same order as for other nitrates, namely 31·2 for barium nitrate, 34·2 for strontium nitrate, $28\cdot2\times10^{-13}$ for lead nitrate, which indicates the approximate validity of Grüneisen's relation for the crystal.

SUMMARY

The principal coefficients of expansion have been determined for the orthorhombic crystal potassium nitrate for the range of temperature 30 to 100° C. by an X-ray method. The values are $\alpha_{11}(b) = 22\cdot0$; $\alpha_{22}(a) = 23\cdot5$; $\alpha_{33}(c) = 182\cdot6\times10^{-6}$. The direction of the greatest expansion is perpendicular to the plane of the O_3 group and the expansion is nearly constant for all directions in the plane of the O_3 group. The approximate validity of the Grüneisen relation is tested by calculating the compressibility from a knowledge of the variation of the Raman frequency with temperature.

The author wishes to record his deep indebtedness to Professor G. N. Ramachandran for the invaluable suggestions he gave and the keen interest he took in the course of this study.

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STUDY OF THE ANISOTROPY OF COSMIC RAYS WITH NARROW ANGLE TELESCOPES

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Received May 13, 1955

When appropriate corrections to remove terrestrial influences are applied to the daily variation of meson intensity, it shows up the anisotropy of the primary cosmic radiations as viewed by an apparatus fixed to the spinning earth. Early data relate almost exclusively to measurements made with ionisation chambers for study of the various types of time variations of cosmic ray intensity. In these the instruments measure radiation from all directions and are therefore not particularly appropriate for study of the daily variation of intensity. Several later studies have been made with counter telescopes having directional characteristics. However in an effort to secure a high counting rate and good statistical reliability of the results, the telescopes generally subtend large angles and hence most of them ignore the requirements for a specialised study of the daily variation. From available determinations it has been in consequence erroneously assumed by several workers that the meson intensity at sea level does not exhibit a daily variation with an amplitude in excess of 0·2 to 0·4%. The notable exceptions to this way of thinking have been the Japanese group of workers and the group at Physical Research Laboratory, Ahmedabad.

Sekido and his co-workers have drawn attention in a series of papers[1,2,3] to the remarkable differences observed in the nature of the daily variation and its changes during periods of magnetic disturbance when measurements are made with vertical telescopes having semi-angles of 12°, 40° and 85°. Having concluded that the daily variation is better observed with narrow angle telescopes they[4] have conducted an experiment with a telescope having a semi-angle of 5° to determine sources of cosmic rays in the galaxy.

About three years ago experiments were commenced at Ahmedabad to determine the result of pushing the technique of narrow angle telescopes to what was considered a practical lower limit. Observations have been made by one of the authors (P. D. B.) with a triple coincidence telescope having a semi-angle of 1·8° in the E-W. plane and a semi-angle of 6·7° in the N-S. plane. The intensity relates to the component of cosmic radiation which

can penetrate a minimum thickness of about 27 cm. of iron at sea level. The experiment was conducted from November 1952 to August 1953 and the mean bi-hourly rate was about 35 counts. The percentage mean daily variation is shown in Fig. 1. In view of the large probable error of each bi-hourly deviation, moving averages over three consecutive bi-hourly values

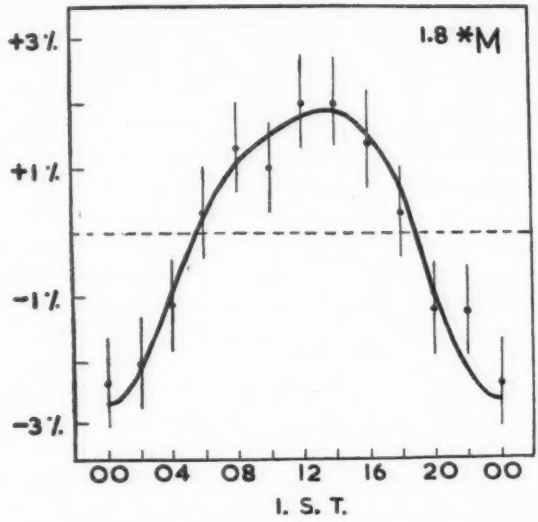

FIG. 1. Daily variation measured with $^{1.8}$T.

have been taken. Harmonic analysis of the daily variation given by the bi-hourly percentage deviations from mean, before taking moving averages, shows that the diurnal amplitude M^D is $2{\cdot}2 \pm 0{\cdot}5\%$ and occurs at noon. The semidiurnal component with an amplitude of $0{\cdot}4 \pm 0{\cdot}5\%$ is not significant but has a negative correlation of $-0{\cdot}81$ with the semi-diurnal component of the daily variation of barometric pressure.

The large amplitude of the daily variation found in this experiment with a narrow angle telescope has prompted us to make a more elaborate study involving simultaneous measurements with telescopes of different angles. It was hoped thereby to determine with some precision the profile of the anisotropy of primary cosmic radiation and also to establish an optimum experimental technique for study of the daily variation of meson intensity.

In Fig. 2 we show the arrangement of counters in an experiment set up by one of the authors (N. W. N.). The apparatus furnishes triple coincidences from three independent telescopes $^{2\cdot5}$T having semi-angles of $2{\cdot}5°$, from two

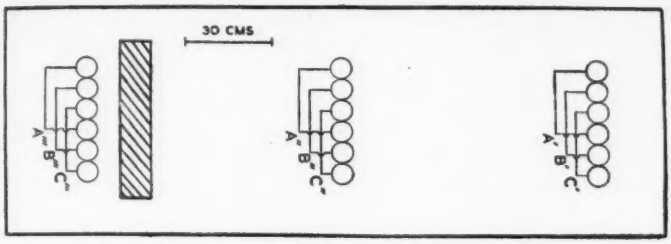

FIG. 2. Experimental Arrangement.

$^{2\cdot5}$T —A′ A″ A‴, B′ B″ B‴, C′ C″ C‴.
^{5}T —(A′B′) (A″B″) (A‴ B‴), (B′C′) (B″C″) (B‴C‴).
^{15}T —(A′B′C‴) (A″B″C″) (A‴B‴C‴).
^{90}T —(A‴B‴C‴).

telescopes ^{5}T with semi-angles of 5° and from one telescope ^{15}T with a semi-angle of 15° in the E-W. plane. All telescopes have semi-angles of 19° in the N.-S. plane. In addition, the total rate ^{90}T from a battery of counters has been taken to give the omni-directional intensity. 12 cm. of lead has been used as shielding in every case. To increase the counting rate of telescopes without change of semi-angle, duplicate sets of counters are placed horizontally displaced from the first set, but connected with them in parallel. This introduces no error except for a negligible contribution from penetrating air showers.

In Table I are given details of the telescopes, their characteristics and period of operation. Since it is known that the daily variation of meson

TABLE I

Details of Telescopes

Telescope	$^{2\cdot5}$T	^{5}T	^{15}T	^{90}T
Daily Variation	$^{2\cdot5}$M	^{5}M	^{15}M	^{90}M
Semi-angle of Telescopes ..	2·5°	5°	15°	90°
No. of Telescopes ..	6	4	1	1
Period for which data available	1954 Jan.-May. Sep.-Dec. 1955 Jan.-March	1954 Jan.-May Sep.-Dec. 1955 Jan.-March	1954 Sep-Dec. 1955 Jan.-March	1954 Jan.-May Sep.-Dec. 1955 Jan.-March
Mean bi-hourly counting rate per Telescope	199	251×4	448×16	499×1024

intensity can alter with the passage of time, it is obviously necessary to compare the performance of the different telescopes over a period when all instruments have simultaneously been working. For such a period we show in Fig. 3 the mean percentage daily variations $^{2\cdot5}$M, ^{5}M, ^{15}M and ^{90}M, superposing the data of as many similar telescopes as have been simultaneously in operation. Figures 3 *a* and 3 *b* relate respectively to the daily variations before and after application of a barometric pressure coefficient $\beta = -2\cdot2\%$ per cm. of Hg. The amplitudes and the times of maxima of the diurnal and the semi-diurnal harmonic components of the daily variations measured with different arrangements are given in Table II. The diurnal components are also shown on harmonic dials in Fig. 4. The values before and after correction for the daily variation of barometric pressure are shown alongside of each other, the former being indicated by asterisks.

The following features emerge from a comparison of the daily variations of meson intensity observed with the different types of instruments.

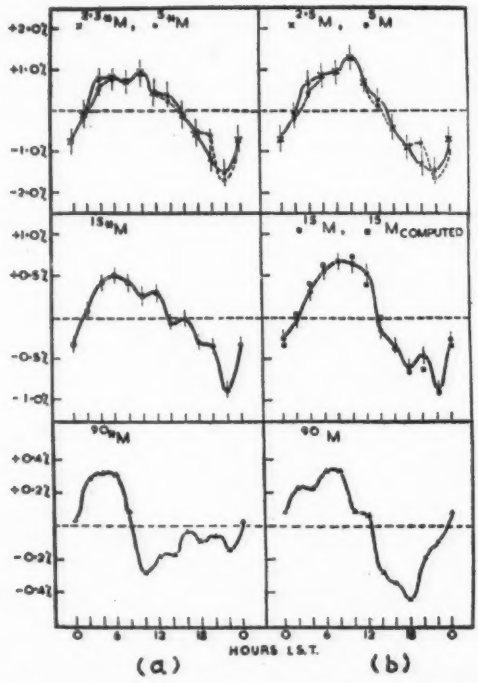

FIG. 3. Daily variations measured with $^{2\cdot5}$T, ^{5}T, ^{15}T and ^{90}T. Figures (*a*) and (*b*) relate respectively to values before and after applying correction for daily variation of pressure.

TABLE II

Percentage amplitudes and the times of maxima of the 1st and the 2nd harmonics of the daily variation measured with different telescopes

Daily Variation	$^{2.5°}$M	$^{2.5}$M	$^{5°}$M	^{5}M	$^{15°}$M	^{15}M	$^{90°}$M	^{90}M
Standard Error	±0·15%	±0·15%	±0·08%	±0·08%	±0·05%	±0·05%	±0·007%	±0·007%
M^D	1·05%	1·21%	0·96%	1·15%	0·44%	0·68%	0·21%	0·31%
$M\phi^D$	134°	133°	135°	134°	122°	121°	50°	80°
M^S	0·32%	0·13%	0·28%	0·06%	0·20%	0·02%	0·16%	0·06%
$M\phi^S$	114°	84°	133°	154°	133°	−82°	135°	−53°
γ	−0·950	..	−0·992	..	−0·999	..	−0·998	..
β	−0·30	..	−0·20	..	−0·20	..	−0·16	..

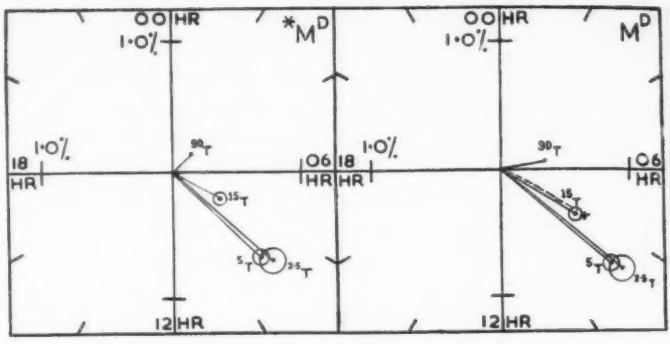

FIG. 4. Harmonic dials showing diurnal components of $^{2\cdot5}$M, ^{5}M, ^{15}M and ^{90}M before (L.H.S.) and after (R.H.S.) applying correction for daily variation of pressure.

1. $^{2\cdot5}$M and ^{5}M are almost identical. They differ markedly from ^{15}M and again from ^{90}M.

2. The percentage amplitude of the diurnal component $^{2\cdot5}M^{D}$ is slightly greater than $^{5}M^{D}$, but not significantly so. However the amplitude which is about 1·2% for the narrow angle telescopes, decreases rapidly for wider semi-angles beyond 5°, and is only about 0·25% for the omnidirectional intensity.

3. The time of maximum $M\phi^{D}$ of the diurnal component corresponds to about 0900 hours for $^{2\cdot5}$M and ^{5}M but becomes progressively earlier with wider semi-angles. $^{90}M\phi^{D}$ corresponds to about 0500 hours.

4. The semi-diurnal component of the daily variation does not alter as much as the diurnal component when measurements are made with the different instruments. There is an indication that the amplitude M^{S} decreases with increasing angle of telescope, but not to the same extent as M^{D}. The time of maximum is fairly constant.

5. There is high negative correlation between the semi-diurnal variations of meson intensity and of barometric pressure, the omnidirectional intensity having the highest correlation. The application of a uniform correction for the daily variation of barometric pressure using a coefficient $\beta = -2\cdot2\%$ per cm. of Hg, results in a virtual elimination of the semi-diurnal components in $^{2\cdot5}$M, ^{5}M and ^{15}M. However ^{90}M is still left with a significant semi-diurnal component.

The choice of $\beta = -2\cdot2\%$ per cm. of Hg is dependent on considerations described elsewhere,[5] but it is not clear whether the application of the same coefficient to ^{90}T as well as to the narrower telescopes is fully justified.

It will be observed that in $^{2\cdot5}$T and ^{5}T, the barometric correction with a coefficient of this magnitude makes relatively little difference to the daily variation of meson intensity. Hence, there is likely to be no great ambiguity in interpretation on account of the operation of uncertain meteorological influences.

Computation shows that meson intensity incident on ^{90}T from directions inclined less than 5° from the vertical, makes a contribution of less than 1·0% to the total omnidirectional counting rate. Thus ^{90}M with a diurnal amplitude of 0·24% represents the integrated daily variation of intensity incident from all directions, and is little affected by the special features of the daily variation characteristic of intensity in directions close to the vertical. On the other hand $^{2\cdot5}$M and ^{5}M indicate that the intensity which exhibits a large daily variation of diurnal amplitude 1·2% is confined almost exclusively to a narrow vertical cone of incidence with a semi-angle of about 5°.

If we assume that intrinsically there are two types of daily variations, the V-type characteristic of the vertical intensity and the O-type characteristic of the omnidirectional intensity, then we have V $\approx$ ^{5}M $-$ ^{90}M and O $\approx$ ^{90}M. These are shown in Fig. 5. ^{15}M is compatible with a superposition of the two types with appropriate weightage factors. This is clear from Figs. 3

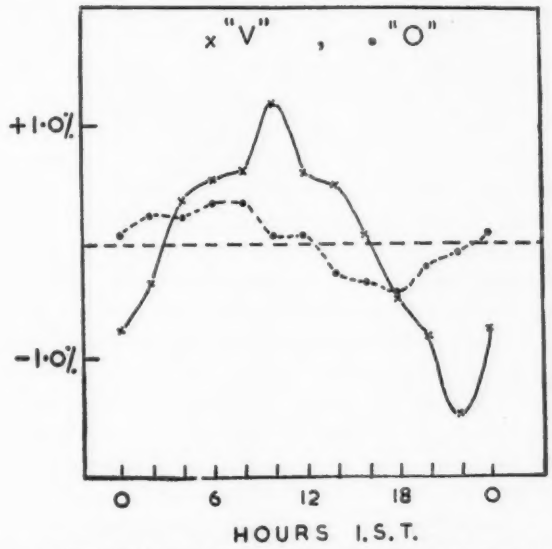

FIG. 5. The V and the O types of daily variations of meson intensity.

and 4 where we show the experimental and computed values of ^{15}M and its diurnal component respectively. The weightage factors have been evaluated from considerations of the geometry and dimensions of ^{5}T, ^{15}T and ^{90}T.

It is remarkable that there is sharp attenuation of intensity showing V-type of daily variation for incidence at angles exceeding 5° with the vertical. Simpson *et al.* have reported for local production of neutrons a daily variation comparable in amplitude to V. Since the nucleonic component of cosmic radiation, which is responsible for the local production of neutrons, gets rapidly attenuated with increasing zenith angle, the neutron monitor essentially measured vertical intensity. Thus a large amplitude of the daily variation appears to be a characteristic of a radiation, which can only penetrate to low levels of the atmosphere at small zenith angles. The main difference in amplitudes of the daily variations[6] observed with Simpson's neutron monitor and the Sitkus ionisation chamber could possibly arise from considerations of directional sensitivity of the measuring instruments rather than from difference in mean energies of primaries to which they respond.

It is important to understand the O variation and its origin. This is dependent on study with narrow angle telescopes pointing towards directions making different angles with zenith in the E.-W. plane. If, as appears likely, the O variation does not arise due to an integration of daily variations vastly different in character for different zenith angles towards east and west, we might be here dealing with a general modulation of the cosmic ray intensity. This would presumably occur near the top of the atmosphere but be under solar control.

Narrow angle telescopes open up a wide new field for experimental investigation of the anisotropy of primary cosmic rays. With low counting rates, data have to be averaged over an extended period to get significant results. Since the nature of the anisotropy changes, often radically, with passage of time, the averaging of the daily variations for several days is rather unsatisfactory. We can however, get around this difficulty by having simultaneous measurements with an array of several independent narrow angle telescopes. Work is now in progress along these lines and a report will be published separately.

We have benefited by discussions with Dr. R. P. Kane. We gratefully acknowledge support of the Atomic Energy Commission of India to our project.

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THE OCCURRENCE OF HIGH MULTIPLE REFLECTIONS FROM THE F2 REGION OF THE IONOSPHERE BASED ON A STUDY OF THE AHMEDABAD RECORDS

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(Communicated by Dr. K. R. Ramanathan, F.A.SC.)

Introduction

Shortly after the installation of the automatic ionospheric recorder at Ahmedabad in January 1953, it was observed that on certain nights groups of pulses other than the regular ones moved swiftly across the time base on the monitor oscilloscope. On the P′-*f* records, they produced traces which started from below the ground pulse and crossed the various P′-*f* traces. Examples of such records are shown in Fig. 1. After the first three months of regular P′-*f* recording, the phenomenon was not observed for some months, but reappeared clearly in December 1953 and January 1954. A study of the phenomenon was then undertaken.

General Features of the Phenomenon

The following features were noticed about these abnormal traces: (1) They occurred only during night. (2) At any frequency the separation in height between two adjacent abnormal traces appeared to be the same as the virtual height of the first reflection trace at that frequency. (3) They ended at the ordinary critical frequency f_0F_2 of the F_2 layer, and on some occasions, there was another group of pulses ending at the extraordinary critical frequency f_xF_2 of the F_2 layer. (4) The ratio of the slope of an abnormal trace to that of the first reflection trace was found to be the same at any frequency and increased in steps of one from one trace to the next. It was therefore concluded that these abnormal traces represented high order reflections from the ionosphere.

The equipment at Ahmedabad is a British N.P.L. type automatic ionospheric recorder and the pulse repetition frequency is 50 per second. If the delay time of a high order multiple reflection is more than 1/50 second, it will be recorded in the second sweep of the time base, and the equivalent height of the multiply reflected trace will exceed 3000 km. (= 1/50 second). The order of reflection can be determined by dividing the slope or the equivalent height of the trace at any frequency by the corresponding slope or height of the first reflection trace at that frequency.

During regular recording, the duration of the time base is usually adjusted to record echoes of equivalent height upto 1000 km. only. However, the time base duration can be extended to record echoes of equivalent height upto about 2000 km. With such an extended time base, multiples upto the 17th order have been recorded. It was considered that it would be worthwhile to increase the height range still further and arrangements were made on a few occasions to record the echoes in two steps, first showing the reflections upto 1500 km., and then delaying the start of the time base so as to record heights between 1500 and 3000 km. Such a double record is shown in Fig. 2 where all multiple echoes from the first to the ninth can be seen.

An examination of the literature showed that high multiple reflections have been noticed by some previous workers on P′-*t* records (Pierce and Mimno,[1] Baird[2]). They attributed these to the curved shape of the ionosphere or humped ionisation contours in the ionosphere. These humps produce a mirror-like converging action on the reflected radio beam. Gipps, Gipps and Venton[3] also explained some abnormal traces in P′-*f* records in terms of humps and concavities in the ionosphere. Munro[4] described certain complexities observed on ionospheric records as the effects of the curvature of the ionospheric surface. Uyeda and Nakata[5] have recently described similar abnormalities observed on *h′-t* and *fct* records and explained them on an assumed inclination of the ionosphere and its movement.

Analysis of Ahmedabad Records Showing High Order Multiple Reflections

A detailed analysis was made of the records showing such abnormal traces and the present note contains a summary of the findings.

The parameters of the F_2 layers were determined by the method of Appleton and Beynon.[6] The virtual heights of the layer at various frequencies were plotted against the corresponding values of Booker and Seaton's function[7]

$$\phi\left(\frac{f}{f_c}\right) = \frac{1}{2}\,\frac{f}{f_c}\,\log_e\left|\frac{f_c+f}{f_c-f}\right| - 1$$

where f_c is the critical frequency of the layer and f any frequency less than the critical frequency. It was found that a neat straight line could be drawn through the points showing that the distribution of electrons in the F_2 layer during the night time was approximately parabolic. The values of the virtual heights at $\phi = -1$ and $\phi = 0$ gave respectively the true height of the base h_0F_2 and the height of the level of maximum ion density h_pF_2, the semi-thickness being given by $h_p - h_0$.

f Mc/s	ϕ	h_2 km.	$h_2/2=h_1$ km.	$h\,(n_1)$ km.	$hn_1/11$ km.	$h\,(n_2)$ km.	$hn_2/12$ km.	$h\,(n_3)$ km.	$hn_3/13$ km.	$h\,(n_4)$ km.	$h\,(n_4)/14$ km.	Mean h_1 km.
1·9	−0·52	445	223	..	..	..	..	..	..	..	..	223
2·0	−0·47	452	226	..	..	..	..	..	..	..	..	226
2·1	−0·37	460	230	..	..	..	..	..	..	..	..	230
2·2	−0·32	470	235	..	..	..	..	3000	231	..	..	233
2·3	−0·21	480	240	..	..	..	..	3090	238	..	..	239
2·4	−0·12	490	245	..	..	2900	242	3160	243	..	..	243
2·5	0·00	500	250	..	..	2970	248	3250	250	3500	250	250
2·6	+0·14	520	260	..	..	3080	257	3350	258	3640	260	259
2·7	+0·33	540	270	2900	264	3200	267	3450	265	3750	268	267
2·8	+0·54	560	280	3080	280	3360	280	3575	275	..	..	280
2·9	+0·95	615	308	..	..	..	..	..	..	..	..	308

The analysis of a typical record (Fig. 1 *b*) is given in the following table. The value of f_0F_2 was 3·0 Mc/s. $h(n_1)$, $h(n_2)$, $h(n_3)$ and $h(n_4)$ are the virtual heights of observed multiples and are found to be nearly equal to 11, 12, 13 and 14 times h_1. h_pF_2 was found to be 250 km., h_0F_2 to be equal to 198 km. and the semi-thickness of the layer was 52 km.

With a view to find out what ionospheric features were associated with high multiples, the parameters h_0, h_p and T obtained from the hourly ionospheric records during the occurrence of the phenomenon were examined.

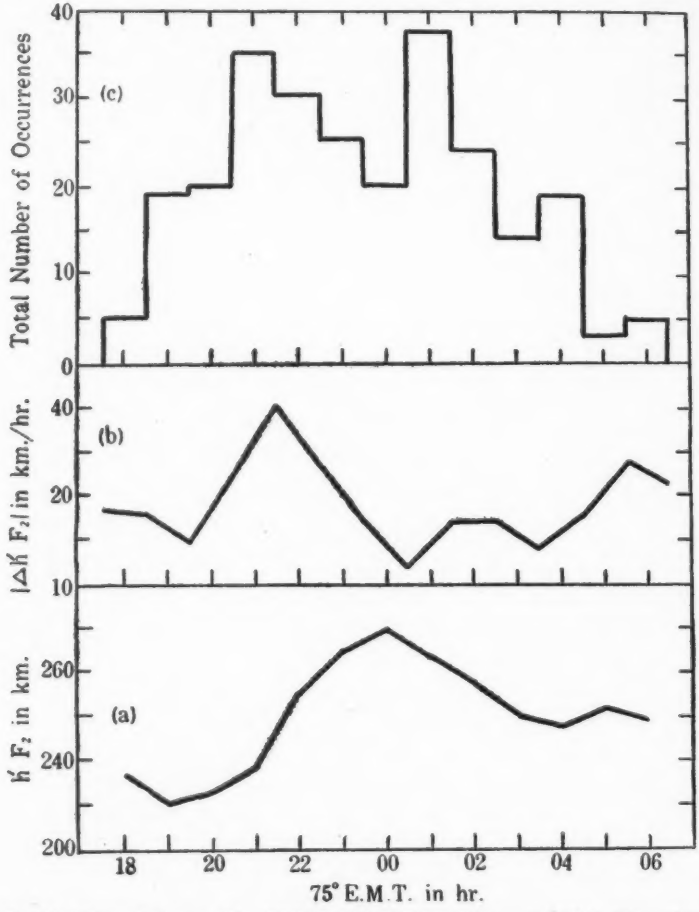

FIG. 3. (*a*) Mean minimum virtual height of the F_2-layer (January 1953 to December 1954). (*b*) Average hour to hour change in $h'F_2$ (January 1953 to December 1954). (*c*) Total number of occurrences of high multiples (January 1953 to December 1954).

In the records of the period January 1953 to December 1954 there were 256 such occurrences. It was found that in general high multiple reflections were associated with a hump in the ionospheric surface over the place of observation. High multiples occurred more or less continuously for many hours in certain periods of the night in some months. Near the critical frequency the multiples are generally weaker, due no doubt to increased absorption as well as to increased slope of the P′-*f* traces.

The distribution of the occurrence of high multiples with the hour of the night is shown in Fig. 3 (*c*). In Fig. 3 (*a*) and 3 (*b*) are shown the mean

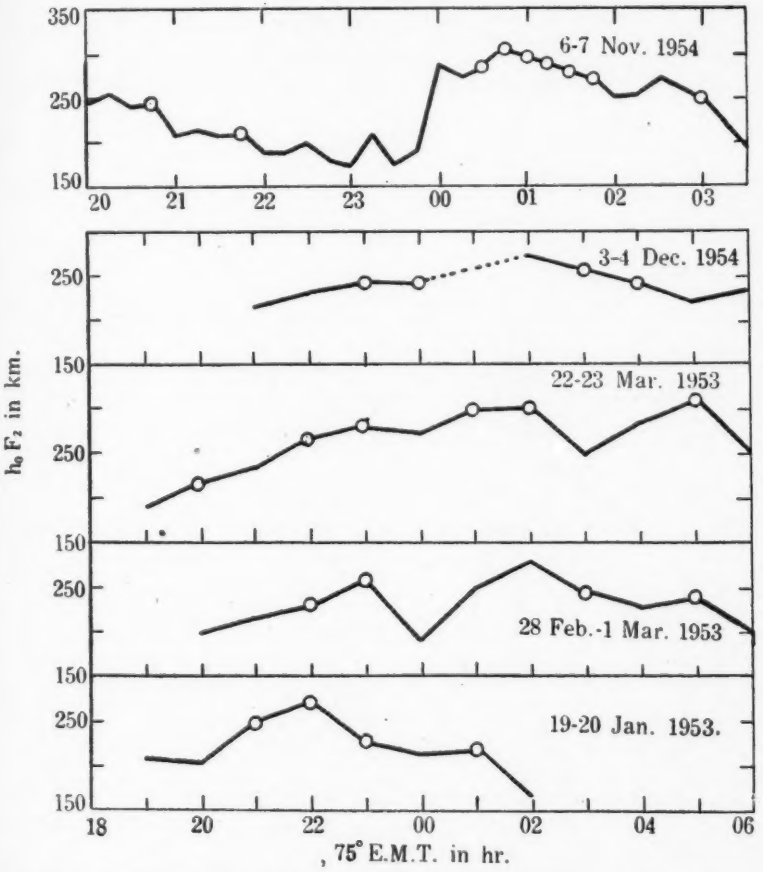

FIG. 4. Variation of h_0F_2 and the occurrence of high multiples on a few nights.

virtual height ($h'F_2$) and the average hour-to-hour change in virtual height ($h'F_2$) during the period January 1953 to December 1954.

It will be seen that the hour of maximum occurrence of high multiples are round about 21 hours, 01 hour and 04 hour. The times of maximum frequency of occurrence of high multiples nearly coincide with or precede the times of maximum rate of change of $h'F_2$. The strength of the peak at 01 hour in Fig. 3 (*c*) is abnormal due to the large number of occurrences of high multiples at that hour in one month only, namely in November 1954.

Figure 4 gives the analysis of the heights of the F_2 layer on a number of individual nights at times when high multiples were observed. The circles represent the times at which high multiples were observed.

Observations at short period intervals taken on a few nights showed fair correlation between the highest order multiple and the curvature of the

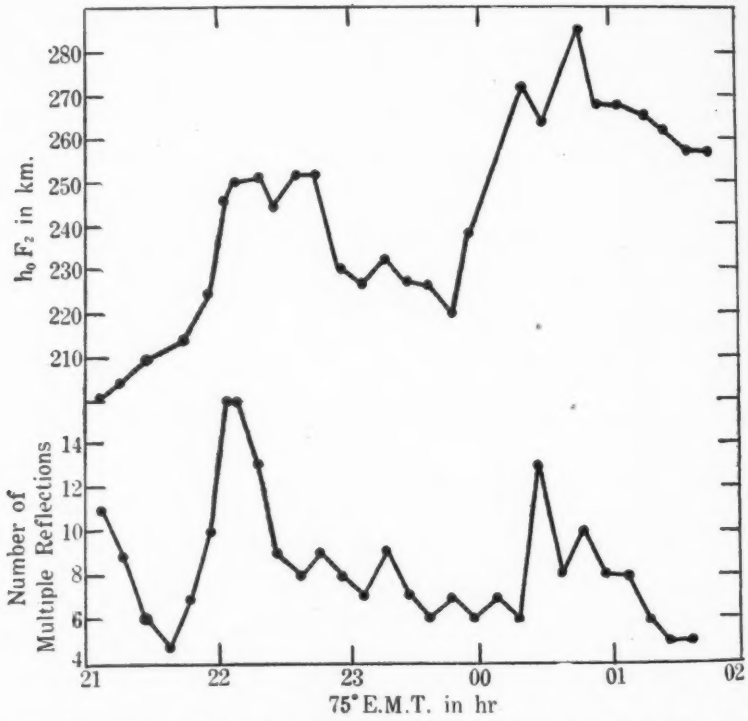

FIG. 5. The variation of the total number of multiple reflections and the height of base of the F_2 layer on the night of 10 to 11 March 1955.

ionospheric surface. Figure 5 shows the results of observations taken on the night of 10–11 March 1955.

It was found that sporadic E does not affect the high multiples except when it blankets the normal reflections. Sometimes high multiples of F_2 and sporadic E can be seen simultaneously recorded over the same frequency range (Fig. 1*b*).

1 F scatter usually reduces the number of multiple reflections but there are some exceptions to this also (Fig. 1 *c*).

Seasonal Variation of Multiple Reflections

It appears from Fig. 6 that high multiples are least frequent in the summer months May to July and most frequent in November to March or

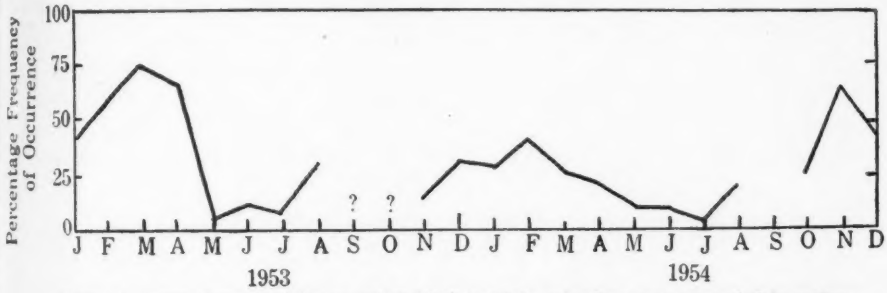

Fig. 6. Percentage of nights on which high multiple reflections were recorded in various months January 1953 to December 1954.

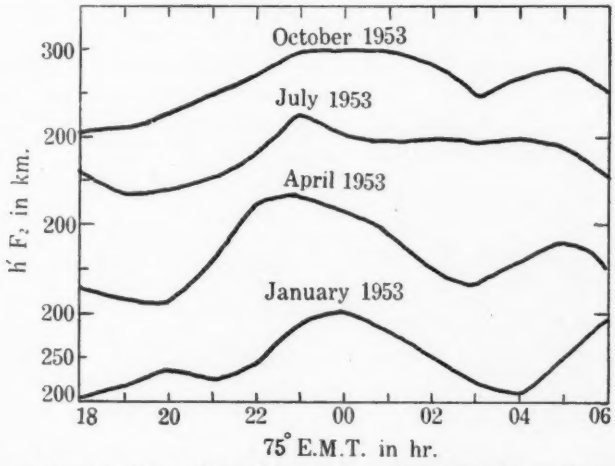

Fig. 7. Variation of $h'F_2$ during night time in different months of the year.

April. This accords with the fact that the daily variations of $h'F_2$ are least pronounced in the summer months and most pronounced in autumn and spring. During summer, E_s and 1 F—scatter are rather more frequent and these no doubt contribute partly to the rarity of high multiples in that season.

IRREGULARITIES IN THE INTENSITIES OF THE MULTIPLE REFLECTIONS

A phenomenon related to the above is the anomalous and varying distribution of intensity of multiple echoes. This phenomenon can be seen clearly on many occasions on the monitor oscilloscope and generally occurs on multiples higher than the third. It is difficult to determine quantitatively intensity ratios of the various multiples from the P'-*f* records but approximate estimates of their intensity can be obtained in some cases.

At times, depending upon the position of the hump with respect to the place of observation, a higher multiple may be formed by combination of waves leaving the transmitter at different angles and arriving at the receiver after being reflected from off-vertical points in the ionosphere. The intensity of some particular multiple may then be greater than that of the lower ones depending upon the curvature, height, and reflection coefficient of the F_2 layer, as well as on the radiation patterns of the transmitting and receiving antennæ.

CONCLUSION

These observations show clearly that even at night, the F_2 layer can by no means be considered as a simple plane reflecting surface and that there are dynamic changes going on in it most of the time. Determination of the reflection coefficient of the layer and of the absorption of the atmosphere below it is rendered particularly difficult because of these changes.

ACKNOWLEDGEMENT

The author is indebted to Prof. K. R. Ramanathan for his keen interest and guidance in the work. The Council of Scientific and Industrial Research (India) is supporting the ionospheric station with financial assistance.

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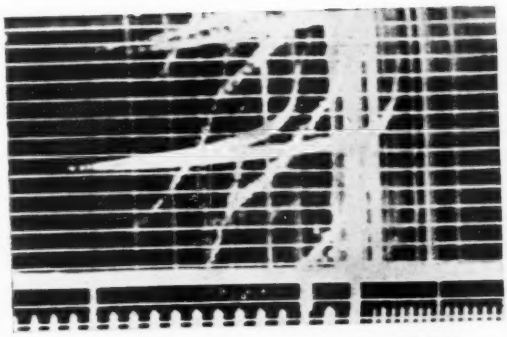


FIG. 1(*a*). P′-*f* record at 01 hr. on 28–10–1954.

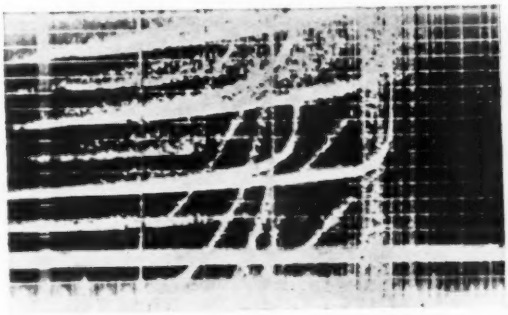


FIG. 1(*b*). P′-*f* record at 02 hr. on 31–1–1953.

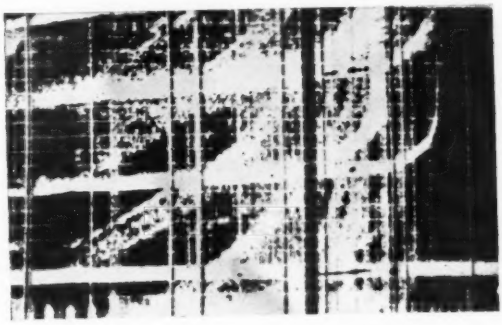


FIG. 1(*c*). P′-*f* record at 00 hr. on 17–3–1953.

FIG. 1. P′-*f* records showing abnormal traces crossing the lower order reflection traces.

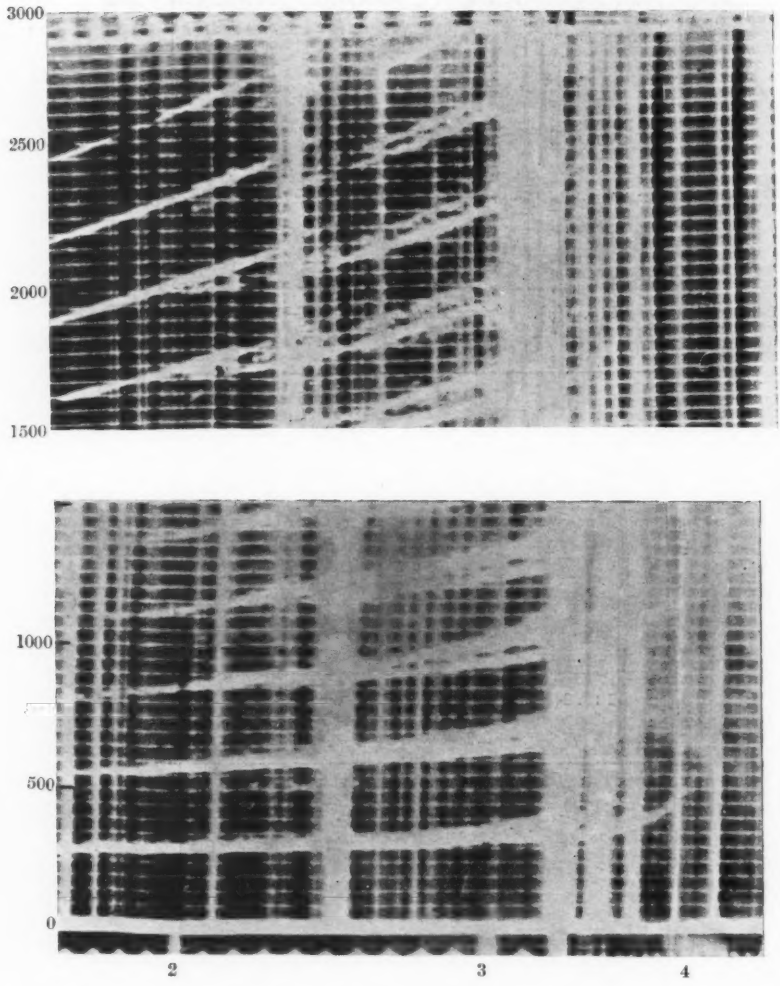


FIG. 2. P′-*f* record at 22 hr. on 10–3–1955 recorded in two steps, one recording echoes upto 1500 km. and the second immediately after, from 1500 km. to 3000 km.

(*a*)

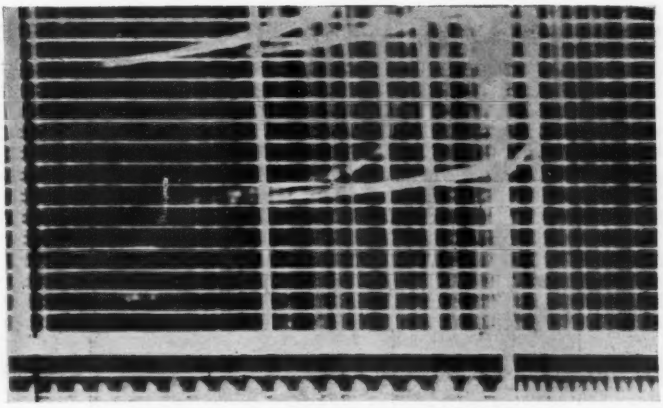


FIG. (*a*) P′-*f* record at 00 hr. on 18-3-1954.

(*b*)

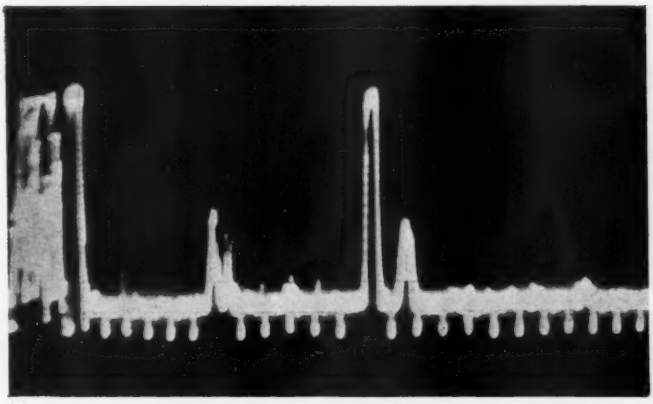
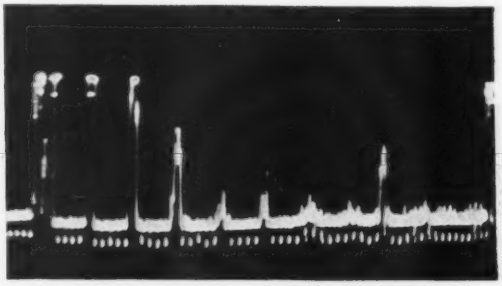
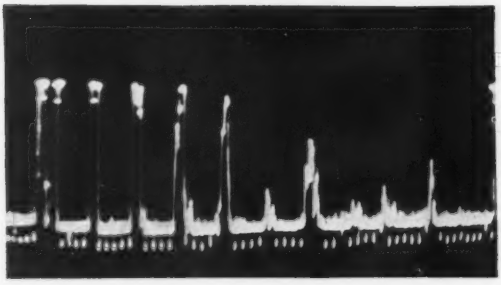


FIG. (*b*) Photo at 2230 hr. on 28-3-1955.

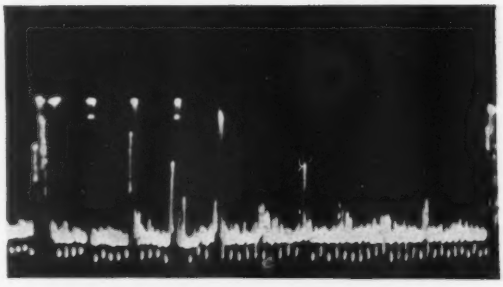
FIG. 8. Records showing second order reflection stronger than the first order reflection



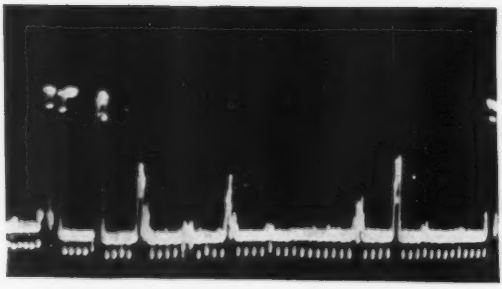
(*a*)
2125 hrs.



(*b*)
2126 hrs.



(*c*)
2127 hrs.



(*d*)
2128 hrs.

FIG. 9. Multiple echoes at intervals of one minute on 24-2-1955.

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